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Irreversible Evolution in Quantum Logics

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The notion of dynamical semigroup is introduced in the quantum logic scheme on the set of the states. Under suitable nonempty mathematical assumptions it is shown that a Heisenberg picture exists equivalent to the Schrödinger one and having many aspects similar to those of the Hilbert case.

1. INTRODUCTION

We reconsider here the problem of the characterization of the time evolution of a physical system in the context of the logic approach to the axiomatics of quantum mechanics. Of course the problem can be solved by specializing the logic scheme to be represented by the Hilbert model and then by defining the usual reversible or irreversible dynamics.

The object of this paper is to characterize the irreversible time evolution which the physical system undergoes directly at the level of the logic scheme, in analogy to what was done for the case of the reversible dynamics by Gorini and Zecca (1975). The definition given there requires essentially that the time evolution, not necessarily linear, preserves the quantum superposition of the states and the orthogonality relation in the logic. Since in general an irreversible time evolution maps decision effects into effects which are not decision effects, the definition and the results in Gorini and Zecca (1972) can be no longer applied.

For this reason we consider here the conventional quantum logic scheme along the line of results of Gudder, Kronfli, and Fisher and Rüttiman. The assumptions are such that the probability measures in the logic are a subset of the Banach space of the states, while the observables are a

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normed linear space because existence and uniqueness of the observables is postulated.

The characterization of the irreversible dynamics of the physical system is proposed in terms of a one-parameter semigroup of linear (not necessarily 1-1) transformations of the Banach space of the states which leaves the set of the probability measures invariant. This is done in analogy to the definition of Kossakowski (1972) for the axiomatic description of non-Hamiltonian time evolutions of physical systems in conventional quantum mechanics.

By using an explicit mathematical request on the theory it is shown that a corresponding irreversible Heisenberg picture exists and has properties analogous to those of the Hilbert case. By another direct mathematical request it is then shown that the Schrödinger picture is implemented by the Heisenberg one, showing in this way the physical equivalence of the two dynamical descriptions.

Finally, by adapting results contained in Gorini and Zecca (1975), it is shown that the case of the reversible dynamics characterized by a one-parameter group of permutations (not necessarily linear) of the states which preserves the quantum superposition of the states and the orthogonality relation on the logic admits by construction equivalent Schrödinger and Heisenberg solution satisfying our definition.

2. DEFINITION AND ASSUMPTIONS

We associate to the physical system a pair (L, S), where L is a logic, namely an orthocomplemented, weakly modular, σ-complete lattice; S is the set of all probability measures on L which is also separating, namely the set of all maps $s \in [0,1]^L$ such that (i) $s(\emptyset)=0$; (ii) $s(\mathbb{1})=1$; (iii) $s(\bigvee_i a_i)=\sum_i s(a_i)$ $(\emptyset, \mathbb{1} \in L,\ a_i \perp a_k)$ satisfying $s(a)=s(b),\ \forall s \in S \Rightarrow a=b$; O is the set of the observables of L, namely of the σ-homomorphism $x: B(\mathbb{R}) \to L$ [$B(\mathbb{R})$ denotes the Borel subsets of $\mathbb{R}$]. We denote by $\sigma(x)$ the spectrum of an observable x, namely the smallest closed subset $F \subset \mathbb{R}$ such that $x(F)=1$. An observable x is bounded if $\sigma(x)$ is compact. We denote again by O the set of the bounded observables of L and by Q the questions of O, that is, the observables x_a defined by $x_a(\{1\})=a$ and $x(\{0\})=a^\perp$, $a \in L$. The identity observable I is defined to be the observable such that $\sigma(I)=\{1\}$. The point spectrum of x is $\sigma_P(x)=\{\lambda \in \mathbb{R} \mid x(\{\lambda\}) \neq \emptyset\}$. We call effects the elements of the set $E=\{x \in O \mid \sigma(x) \subset [0,1]\}$. The expectation value of an observable when the system is in the state s is denoted by $s(x)=\int_{\mathbb{R}} \lambda s(x(d\lambda))$. It must be remarked that, in contrast to the Hilbert model, in general an observable is not determined by its expectations (Gudder, 1978). Sufficient conditions for the uniqueness are, for instance, that (a) S is strongly order determining and (b) observables having the same expectations have countable spectra or at most one limit point spectrum (Gudder, 1966*a,b*, 1970, 1978).

The sum $z=x+y$ of observables can be defined through the sum of their expectations: $s(z)=s(x)+s(y)$, $\forall s\in S$. However, existence is not in general guaranteed (Gudder, 1970). In the following, uniqueness and existence properties for observables are assumed. Under these assumptions O is a normed linear space with the norm $|x|=\sup\{|\lambda|\,|\lambda\in\sigma(x)\}$ (Gudder, 1966*a*).

For the next considerations it is useful to consider S as a subset of a Banach space. This can be achieved in different ways by introducing suitable metrics in the set of the states as has been done, for instance, in Kronfli (1970*a*), Gudder (1973), and Fisher and Rüttiman (1978, and references therein).

Here the mathematical framework developed by Fisher and Rüttiman (1978) will be adopted. Accordingly, K denotes the cone of the elements $m\in\mathbb{R}^L$ such that (i) $m(\varnothing)=0$, (ii) $m(a\vee b)=m(a)+m(b)$ $(a\perp b)$, and (iii) $m(a)\geq 0, \forall a\in L$; Ω denotes the set of the elements $m\in K$ such that $m(\mathbb{1})=1$ and $V=\operatorname{lin}\Omega=K-K$. A norm (the base norm) can be introduced in V in terms of the Minkowski functional of the convex hull $D=\operatorname{con}(\Omega\cup-\Omega)$

$$\|m\|=\operatorname{Inf}\{d>0\,|\,m\in dD\}$$

For $m\in K$ one has in particular $\|m\|=m(1)$, so that $\|s\|=1$, $\forall s\in S$. The normed space $(V, \|\cdot\|)$ is a Banach space with respect to the base norm provided $\Omega\neq 0$.

The closure X of the linear span of S in V will be called the state space and its Banach dual denoted by X^*. Defining $T(a)(m)=s(a)$, $m\in V$, $a\in L$, then $L_T=\{T(a)\,|\,a\in L\}\subset V^*\subset X^*$ (Fisher and Rüttiman, 1978). The map T has the following properties: (i) T is an injection $L\to X^*$, since S is separating, (ii) $T(\varnothing)=0$, (iii) $\|T(a)\|\leq 1$ and $T(\mathbb{1})=1$ in X^*, since $\|T(a)\|\leq 1$ in V^*; (iv) $T(a)$ is weakly countable additive on L:

$$T\Big(\bigvee_n a_n\Big)(s)=\sum_n T(a_n)(s) \quad \forall s\in X \qquad \text{and} \qquad a_i\perp a_k, \quad i\neq k$$

As a consequence, the results of Kronfli (1970*b*) concerning the integration theory of observables holds. Accordingly, the map $u\to u^*$ given by

$$u^*=\int_{\mathbb{R}} y\hat{u}(dy) \qquad (u\in O)$$

where $\hat{u}$ is the bounded, weakly countable, additive X^*-valued measure on $B(\mathbb{R})$ defined by $\hat{u}(A)(p)=p(u(A))$ $[p\in X, A\in B(\mathbb{R})]$; it maps O into X^*

and is such that $u^*(s)=\int_{\mathbb{R}} ys(u(dy))$ coincides with the expectation value for the observable u when the system is in the state $s \in S$. We denote by

$$B=\left\{\int_{\mathbb{R}} y\hat{u}(dy)\,\middle|\, u\in O\right\}$$

the bounded linear functional on X "represented" by the bounded observables.

3. THE TIME EVOLUTION

We adapt here the definition of dynamical semigroup introduced by Kossakowski (1972).

Definition 1. A one-parameter family $t \to \alpha_t$, $t \geq 0$, of linear endomorphisms of V is said to be a quantum dynamical semigroup of the quantum physical system if: (i) $\alpha_t S \subset S$; (ii) $\|\alpha_t m\| \leq \|m\|$, $\forall m \in X$; and (iii) $\alpha_t \alpha_\tau = \alpha_{t+\tau}$, $t, \tau \geq 0$.

We remark that condition (ii) is a consequence of condition (i). Indeed, if $d>0$ is such that $m \in d$ con$(\Omega \cup -\Omega)$, then by the linearity of α_t and condition (i), $\alpha_t m \in d$ con$(\Omega \cup -\Omega)$. Hence, from the very definition of the norm, $\|\alpha_t m\| \leq \|m\|$. Conditions (ii) and (iii) together with (iv) α_t is a strongly continuous function of $t \geq 0$, and (v) s-$\lim_{t\to 0} \alpha_t m = m$, $\forall m \in V$, are sufficient conditions to ensure the validity of the Hille–Yosida theorem (see, for instance, Yosida, 1971), which implies the existence of the infinitesimal generator of the dynamical semigroup.

In the Hilbert model, where $S=S(H)$, $S(H)$ being the set of all the probability measures on the complete orthocomplemented, weakly modular lattice $L(H)$ of a separable complex Hilbert space H, by the result of Gleason (1957), $S(H)$ can be identified with the convex set of positive trace 1 operators on H and X with $L^1(H)$, the real Banach space of self-adjoint trace-class linear operators on H. In this model, if conditions (i)-(v) are assumed, the definition of dynamical semigroup coincides with the one given by Kossakowski (1972). The generator of a dynamical semigroup has been determined, with the preliminary assumption of complete positivity, in the case of an N-level system [see Gorini *et al.* (1976) and, for an elementary proof, Parravicini and Zecca (1977)] and in the bounded case (Lindblad, 1976). For the unbounded case some results can be found in Davies (1977, 1979). The dynamical semigroup has been widely used in the phenomenological treatment of open systems (Lanz *et al.*, 1971; Haake, 1973; Frigerio *et al.*, 1978).

We now introduce the Heisenberg picture in the logic scheme. According to the results quoted in the previous section, for every observable u and dynamical semigroup $t \to \alpha_t$, one has

$$\|u^*(\alpha_t s)\| \leq k\|s\| \qquad \forall s \in X, \quad t \geq 0$$

for some positive constant k, since $u^* \in X^*$ and α_t is a contractive map. By setting

$$D_t^\alpha(u)^*(s) = u^*(\alpha_t s) \qquad \forall s \in X$$

we have from the definition $D_t^\alpha(u)^* \in X^*$, $\forall t \geq 0$. The expression $u^*(\alpha_t s)$ represents for $s \in S$ the expectation of the observable u when the system has evolved into the state $\alpha_t s$. In order to generalize the Heisenberg picture we assume that there exists a (bounded) observable such that its expectation coincides with the mentioned one when the system is in the state s.

Axiom 1. $D_t^\alpha(u)^* \in B$, $\forall t \geq 0$.

On the basis of Kronfli's result of the previous section, there exists an observable $D_t^\alpha(u) \in O$ such that $u^*(\alpha_t s) = D_t^\alpha(u)^*(s)$, namely

$$\int_{\mathbb{R}} y(\alpha_t s)(u(dy)) = \int_{\mathbb{R}} ys(D_t^\alpha(u)(dy)) \qquad \forall s \in X, \quad t \geq 0$$

According to our assumption of the existence and uniqueness of observables, the map $D_t^\alpha(u)$ is a linear map on O.

Proposition 1. The one-parameter family $t \to D_t^\alpha$ of maps $O \to O$ has the following properties:

(i) $D_{t+\tau}^\alpha = D_t^\alpha D_\tau^\alpha$, $t, \tau \geq 0$.
(ii) $D_t^\alpha(E) \subset E$, $t \geq 0$.
(iii) $D_t^\alpha(I) = I$, $t \geq 0$.
(iv) $|D_t^\alpha(u)| \leq |u|$, $t \geq 0$, $u \in O$.

Proof. (i) This follows from the definition of D_t^α.

(ii) According to results of Gudder (1965, 1970, 1978), denoting $V(x) = \{s(x) \mid s \in S\}$ $(x \in O)$, then $V(x)$ is the small closed interval containing $\sigma(x)$. If $x \in E$,

$$\{(\alpha_t s)(x) \mid s \in S\} \subset [0, 1]$$

and

$$[0, 1] \supset V(D_t^\alpha(x)) = \{D_t^\alpha(x)(s) \mid s \in S\} \supset \sigma(D_t^\alpha(x))$$

(iii) By the results mentioned in (ii), $\{1\} = V(I) = V(D_t^\alpha(I)) = \sigma(D_t^\alpha(I))$. Hence $D_t^\alpha(I) = I$, $\forall t \geq 0$.

(iv) This follows by the same arguments which prove (ii). ∎

The one-parameter semigroup $t \to D_t^\alpha$ represents the Heisenberg dynamical semigroup induced by the Schrödinger dynamical semigroup

$t \to \alpha_t$. The problem arises of finding the condition under which the Heisenberg picture induces the Schrödinger one. We give a condition in order to have the equivalence of the two descriptions. Let $t \to D_t$ be a one-parameter family of linear maps $O \to O$. If $u \in O$, we set $D_t^*(u)(s) = s(D_t(u)) = (\alpha_t s)(u^*)$. From the definition, $\alpha_t s$ is a linear map on X^*.

Axiom 2. $\alpha_t s \in X$, $t \geq 0$, $s \in X$.

Proposition 2. Let $t \to D_t$ be a Heisenberg dynamical semigroup of maps $O \to O$ satisfying conditions (i)–(iv) of Proposition 1. Then if Axiom 2 holds, the corresponding $t \to \alpha_t$ is a Schrödinger dynamical semigroup.

Proof. One has $\alpha_t \alpha_\tau = \alpha_{t+\tau}$ from the semigroup property of D_t. Suppose now $s \in S$. If $x_a \in Q$, one has

$$\int_{\mathbb{R}} y(\alpha_t s)(x_a(dy)) = (\alpha_t s)(a) = \int_{\mathbb{R}} ys(D_t(x_a)(dy)) \in [0, 1]$$

from assumption (ii) of Proposition 1. Moreover, $(\alpha_t s)(\varnothing) = 0$, since D_t is a linear map and $(\alpha_t s)(\mathbb{1}) = 1$. Suppose now $a = \bigvee_i a_i \in L$ $(a_i \perp a_k)$. For the corresponding questions we have $x_a = \sum_i x_{a_i}$ by the definition of sum of observables. Since $x_N \to x$ in O means $s(x_N) \to s(x)$, $\forall s \in S$, and this is equivalent to $x_N \to x$ in the norm of O because $\sup\{|\lambda| : \lambda \in V(x)\} \equiv |x|$ $(x \in O)$ for bounded observables (Gudder, 1965), by assumption (iv) of Proposition 1 we have $D_t x_a = \sum_i D_t x_{a_i}$. Hence

$$(\alpha_t s)(a) = \int_{\mathbb{R}} ys(D_t(x_a)(dy)) = \sum_i \int_{\mathbb{R}} ys(D_t(x_{a_i})(dy)) = \sum_i (\alpha_t s)(a_i)$$

so that $\alpha_t S \subset S$. In turn this implies that α_t is a contraction on X as remarked after Definition 1.

Under the assumptions of Axioms 1 and 2 the mean value of a bounded observable gives the same result when calculated with the Heisenberg as with the Schrödinger picture.

The result (ii) of Proposition 2 shows that the questions [or decision effects (Ludwig, 1974)] are in general no longer stable under the irreversible Heisenberg dynamics, while this property holds for the set of all the effects.

The equivalence of the Schrödinger and Heisenberg dynamics can be shown to exist by construction in the case of the reversible dynamics under slightly different physical assumptions. ■

Proposition 3. Let the logic L be complete, and S be the set of all σ-additive probability measures on L such that, by setting $S_1(a) = \{s \in S | s(a) = 1\}$ $(a \in L)$; the following hold:

(i) $a \leq b \Leftrightarrow S_1(a) \subset S_1(b)$, $a, b \in L$.
(ii) $S_1(\bigwedge_\alpha a_\alpha) = \bigcap_\alpha S_1(a_\alpha)$, $\forall \{a_\alpha\} \subset L$.

Let $t \to \alpha_t$ be a one-parameter group of permutations (not necessarily linear) of S such that:

(iii) $\forall a \in L$, $\exists b \in L$ such that $\alpha_t S_1(a) = S_1(b)$ and $\alpha_t S_1(a^\perp) = S_1(b^\perp)$, $\forall t \in \mathbb{R}$.

Then there exists a one-parameter group $t \to u_t^\alpha$ of automorphisms of L and a one parameter group $t \to \rho_t$ of σ-convex automorphisms of S which is implemented, namely such that $(\rho_t s)(a) = s(u_t^\alpha(a))$, $\forall a \in L$, $t \in \mathbb{R}$.

Proof. The result follows after checking that, with our assumptions, the results in Gorini and Zecca (1975) still hold and then by setting

$$\alpha_t S_1(a) = S_1(u_t^\alpha(a)), \qquad (\rho_t s)(a) = s(u_t^\alpha(a)), \qquad a \in L, \quad t \in \mathbb{R}$$

The assumption (iii) of Proposition 3 is essentially equivalent to the request that the (quantum) superposition of states and the orthogonality relation in L are preserved under time evolution. ■

4. CONCLUDING REMARKS

In this paper a possible characterization of irreversible dynamics has been introduced into the quantum logic scheme by the notion of dynamical semigroup (Definition 1). The Schrödinger-like definition is formulated in a context where the states have been structured to Banach space and existence and uniqueness of observables has been assumed.

The results, namely the equivalence of the Schrödinger and Heisenberg pictures, with partial results analogous to the Hilbert case, are based on the request contained in Axioms 1 and 2, which seem to be unavoidable for getting the physical results.

Besides the problem of the existence and uniqueness of observables, for a critical discussion of which we refer to Gudder (1978), it would be satisfactory to have more remote mathematical assumptions than the explicit request of Axioms 1 and 2 in order to obtain the same results. In particular, a detailed analogy of the duality structure of the trace-class and bounded operators in the Hilbert model would be useful. A problem suggested by the treatment of the reversible dynamics of Proposition 3 is that of checking whether the quantum superposition in its general intrinsic formulation is preserved under a dynamical semigroup, a circumstance which holds for the (linear) dynamical maps in the Hilbert model (Zecca, 1981) and which seems not to be of direct evidence here. In this connection a problem of interest is that of characterizing the transformations (*a priori* not necessarily linear or bijective) of the set of the states which preserve superposition in its intrinsic formulation at the level of the quantum logic scheme. However, this study has problematic aspects (Gorini and Zecca, 1975) and admits solutions not necessarily linear already in the case of the reversible dynamics (Zecca, 1976).

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